The Unit Circle as a Grounded Conceptual Structure in Pre-Calculus Trigonometry

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Abstract (150 words)

Learning trigonometry poses a challenge to many high school students, impeding their access to careers in science, technology, engineering, and mathematics. We argue that a particular visuospatial model called the unit circle acts as an integrated conceptual structure that supports solving problems encountered during learning and transfers to a broader range of problems in the same domain. We have found that individuals who reported visualizing trigonometric expressions on the unit circle framework performed better than those who did not report using this visualization. Further, a brief lesson in use of the unit circle produced post-lesson benefits relative to no lesson or a rule-based lesson, but only for participants who exhibited some partial understanding of the relevant concepts in a pretest. The difficulties encountered by students without sufficient prior knowledge of the unit circle underscores the challenge we face in helping them build grounded conceptual structures that support acquiring mathematical abilities.

Keywords: Visuospatial representation, learning trigonometry, conceptual grounding, learning and transfer

Mickey, K. & McClelland, J. L. (in press). The unit circle as a grounded conceptual structure in Pre-Calculus Trigonometry. In D. C. Geary, D. B. Berch, R. Ochsendorf and K. Mann Koepke (Eds.), *Acquisition of Complex Arithmetic Skills and Higher-Order Mathematics Concepts.* Elsevier/Academic Press.

The United States remains behind other nations in mathematics and science education. According to results from the 2012 assessment by the Program on International Student Assessment (PISA; Kelly et al., 2013), only 25% of U.S. 15-year-olds reached the level where they could understand and complete higher order tasks such as "solv[ing] problems that involve visual or spatial reasoning…in unfamiliar contexts" (OECD, 2004, p. 55), less than the average level (31%) of students across the 34 participating OECD-member countries. Like the National Mathematics Advisory Panel (2008), we believe that our nation must find ways to enhance the training of its high-school students in mathematics and science, so they can have the opportunity to compete for positions in highly STEM-dependent technology fields in industry, government, research, and education.

Our research explores the role of a visuospatial representation that grounds the mathematical concepts covered in pre-calculus trigonometry. A difficult subject for many, trigonometry sits at the gateway to entry into university-level science, mathematics, and technology coursework and ultimately into careers in STEM disciplines. Our research explores whether it may be possible to enhance reliance on a spatially grounded conceptual model, which we refer to as the unit circle, and thereby increase high school students' success in this difficult subject. In preliminary studies reviewed below, we found that individuals who report visualizing trigonometric quantities as measurements within the unit circle perform better than those who do not report using this model. We have also found that a brief lesson in the use of the unit circle produces post-lesson benefits relative to no lesson or a rule-based lesson for some participants, further supporting the view that the unit circle provides a grounded conceptual framework that supports acquiring an understanding of trigonometry.

These promising results are limited, however, in that the participants were Stanford undergraduates who had prior exposure to trigonometry, and those who performed poorly in the pre-test showed little or no benefit. Our proposed research seeks to adapt and extend the lesson to make it more suitable for high school and community college students who are just learning trigonometry for the first time, and to examine whether such a lesson might help promote mastery of essential trigonometry concepts among such students. At the end of this chapter we consider several specific challenges we face in addressing this goal. These include the possibility that students may lack relevant knowledge of mathematical ideas that could be considered to be prerequisites for understanding the unit circle; the possibility that their knowledge of these elements may not be strong enough to allow them to manipulate the elements of the representation internally (mentally) rather than relying on external supports; and finally, the possibility that many students may hold beliefs about the nature of mathematics and/or their own abilities that make it difficult for them to treat trigonometry as a coherent conceptual system to be learned.

GROUNDED CONCEPTUAL STRUCTURES IN MATHEMATICAL COGNITION

Historically, multiple points of view have emerged regarding the nature of mathematical reasoning and of factors that allow mathematical reasoning to be successful (Dantzig, 2007). On the one hand, it is common to view mathematics as an essentially formal system, in which structured arrangements of abstract symbols are manipulated according to structure sensitive rules (Marcus, 2003; Newell & Simon, 1961; Russell, 1903). On the other hand, many have suggested that mathematical and scientific reasoning often operates on idealized objects humans can manipulate in their minds, mirroring external manipulations of real objects (Barwise & Etchemendy, 1996; Harnad, 1990; Lakoff & Núñez, 2000). As one example, Shepard (2008, see Figure 1) described an ancient proof of the Pythagorean theorem that is carried out through the mental manipulation of four congruent right triangles with sides of lengths a and b and hypotenuse of length c, within a bounding square with sides of length $a + b$. In panel C, the region within the bounding square not covered by the four triangles has area $a^2 + b^2$. In D, which can be constructed by translating three of the triangles in C (without changing their shape), the region not covered by the four triangles has area c^2. To individuals who have an intuitive appreciation for conservation of area under translation and who appreciate how area depends on the lengths of sides of right-angled figures, the proof seems intuitively compelling, though care is required to make the proof rigorous.

Shepard's (2008) article makes bold claims about the role of spatial reasoning in mathematics and even in scientific discovery, and there is now a body of evidence to support these claims. Wai, Lubinski and Benbow (2009) present evidence supporting the claim that spatial ability represents a third psychological attribute over and above verbal and mathematical ability, and is strongly associated with success in the physical sciences, mathematics, and engineering. Adolescents with greater spatial ability are more likely to major in a STEM field, to enter a STEM occupation, and to produce STEM publications and patents (Kell, Lubinski, Benbow and Steiger, 2013). A broad range of other work, reviewed in a meta-analysis by Uttal et al. (2013), supports the view that spatial ability can be taught. This work suggests that individuals across the spectrum of spatial ability can benefit to a comparable extent, and that the typical extent of benefit from the interventions considered was equal to a shift of nearly half of one standard deviation of spatial ability scores in the whole population.

The work just described has been construed as supporting a perspective in which there is some generalized *visuospatial faculty*, perhaps malleable but quite general – or, if not fully general, its partitioning is still based on a broad distinction, such as a distinction between within-object and between-object processes (Uttal et al., 2013). An alternative perspective draws on insights from seminal work in cognitive science, developmental psychology, and the learning sciences on the role of an acquired understanding of a set of integrated conceptual relationships that, in many domains of mathematics, can be represented by externalized visual depictions like those in Shephard's proof. This perspective contrasts with the symbolic perspective (i.e., that we manipulate expressions using structure sensitive rules) and with the visuospatial faculty idea. We emphasize this third perspective, and focus on helping students acquire an integrated understanding, while acknowledging that both mastery of the rules of symbol manipulation and generalized visuospatial ability may also have roles to play in mathematical cognition.

For us, the essential idea in what we will call the *grounded conceptual structure* perspective is that symbolic expressions – be they sentences or equations – need not be manipulated simply as such. Instead, we argue that they can (and should) be used to construct internal representations in which tokens standing for objects referred to in the expressions are placed in particular relationships with each other (Bransford, Barklay, & Franks, 1972; Glenberg et al., 2004), within a conceptual framework that the comprehender understands (Bartlett, 1932; Bransford & Johnson, 1972). Consider the passage 'Ben needs to feed the animals. He pushes the hay down the hole. The goat eats the hay." (from Glenberg et al., 2004). Comprehension of this vignette is thought to involve mentally acting out Ben's and the goat's activities. This process can be supported by manipulation of toy objects corresponding to those mentioned in a play farm set, or it can occur entirely in the imagination. Either way, implied relationships (Ben is above the goat) may be captured in the constructed result. A similar visuospatial construction can occur when a person understands the expression $a^2 + b^2 = c^2$. The symbol manipulation approach to conceptualizing mathematical ability fails to make contact with these ideas, and leaves the student performing calculations according to arbitrary memorized rules rather than constructing representations of meaningful quantities that can be connected to the properties of referenced or imagined objects in the world (Thompson et al., 1994).

When it comes to mathematical expressions, they may in some instances be constructed with reference to very specific situations. Thus we may convert 'Together Ann and Ben have 7 apples. If Ben has 3 apples, how many apples does Ann have?' into expressions such as $a + b = 7$ and $b = 3$. Purely symbolic actions could then be carried out, but mathematics educators have argued persuasively that such actions are highly error prone. Instead, maintaining contact with the referenced context deepens engagement with the underlying relationships (Thompson et al., 1994), helps prevent errors (Mayer & Hegarty, 1996), and facilitates transfer (Lewis, 1989) by providing a specific conceptual grounding from which a more generalizable representation of the meanings of the mathematical expressions may ultimately emerge, or at least, to which a more generalizable representation may be linked, as discussed more fully below.

In the framework just described, a visuospatially grounded representation might provide a coherent conceptual framework within which statements in a particular mathematical domain “make sense”. In this view, articulated by Case (1996), the visuospatial representation participates (along with other elements) in providing an abstract, generalizable, framework that can be applied to a wide range of specific instantiations – constituting what he calls a *core conceptual structure*. Case (1996) builds up these ideas for natural number (his ‘mental number line’) and extends this to encompass fractions (Moss & Case, 1999). The representation is abstract and generalizable, because (in the case of the number line) it can be applied to sets of enumerable items of any kind, be they coins, toy frogs, or even items of a less tangible nature (days of the week, even ideas).

The visuospatially grounded conceptual representations we propose are related in some ways to other proposals in the psychological literature that are posed as alternatives to symbol manipulation, but are different is some important ways. Shepard (2008; Shepard & Chipman, 1970) and Dehaene (1992) discuss visuospatial representations in mathematics but do not emphasize the role of learning or of culturally-defined conventions in structuring such representations. The mental structures we describe may also be like Johnson-Laird’s (1983) mental models in some ways, in that they involve internal representations that capture features of and relations among entities that can have external referents. One difference is that in our case we are specifically interested in representations with visuospatial structure, whereas Johnson-Laird focuses primarily on representations that do not have an explicitly spatial character. Also, we emphasize the role of extensive exposure to a culturally constructed system that may adhere to a set of partially arbitrary conventions, usually acquired in a formal educational setting. These different perspectives may be mutually compatible, but the emphasis is certainly different.

Several issues for learning and learners arise in our framework. One of these is ***gradual integration and elaboration.*** Acquisition of a fully elaborated core conceptual structure is a process that occurs gradually over time, in part because of a gradual associative process that interlinks relevant parts and in part because the full structure may depend on previous consolidation of constituent structures. For example, an initial simple conceptual structure for numbers from 1-10 may be elaborated to provide the basis for a more complex conceptual structure for the numbers from 1-100 by using one copy of the structure for the ten decades and embedding ten additional copies to represent the numbers within each decade. The ability to create the linkages is thought to depend on the gradual consolidation and integration of the representation of the constituent structures, groups of 10 in this case. Furthermore, a range of factors may play a role in fostering the acquisition and construction of a particular conceptual structure or the engagement of relevant background knowledge that could contribute to supporting performance while the structure is developing.

One such issue that we will consider is ***externalization then internalization.*** Based on the findings of Glenberg et al. (2004) in their research described above, requiring learners to engage in the actual manipulation of a set of physical toys to act out the content of a symbolic expression (‘Ben needs to feed the animals. He pushes the hay down the hole’), then gradually removing this requirement while instructing the learner to imagine the externalized representation while reading other similar texts could help establish the practice of mapping symbolic expressions to their meaning in the conceptual system represented by the diagram, rather than relying on surface representations.

Another potentially important issue is ***generic vs. specific content.*** The unit circle representation that we introduce below is generic in that it can be applied to a wide range of specific contexts; relations that hold in this generic model also hold in a range of specific instantiations to which the model can then be applied. As Kaminski et al. (2008) have argued, this may have advantages over models of specific systems that can bring additional irrelevant content that may obscure the relationships that the model is intended to convey. While there may be benefits of starting with a specific familiar instance of a content domain and then shifting to the more abstract/generic unit circle (Goldstone & Son, 2005), this could complicate our effort to compare lessons based on the unit circle to rule-based lessons. We have, therefore, chosen to stay with the abstract unit-circle framework in our initial investigations.

The Unit Circle as a Grounded Conceptual Structure for Trigonometry

The above ideas provide the setting for our exploration of the role of a particular visuospatial representation in trigonometry. Specifically, we hypothesize that a particular visuospatial representation – the unit circle – links trigonometric expressions to measureable properties of a physically realizable visuospatial model that learners can internalize and then use as a basis for reasoning about relationships between trigonometric expressions that they encounter in pre-calculus trigonometry. The topics and ideas encompassed under this heading build on prior exposure to the concepts of sine, cosine, and tangent as these are encountered in right triangles (sometimes called 'right-triangle trigonometry'), extending their definitions and introducing additional trigonometric functions. The topics include relationships among these functions, graphs of these functions, rules for converting among expressions for such functions, and applications of such functions to problems throughout the physical sciences and engineering.

A reader might be forgiven for thinking that the domain of trigonometry would be one in which, if any, spatial thinking and visualization would be emphasized. Trigonometry appears at first glance to be predominantly visuospatial in nature: Standard definitions of the three elementary trigonometric functions are usually given in terms of ratios of lengths of sides of right triangles, which can be drawn on paper or even constructed out of wood. Yet, an examination of textbooks currently in use in pre-calculus trigonometry courses and of the articles we have found on the understanding and teaching of trigonometry presents a fragmentary and in some ways inconsistent picture. Many texts (e.g., Foerester, 1990; Hornsby et al., 2011, recently-used texts at Palo Alto High School and Foothill College, respectively) emphasize graphing and the use of the unit circle as a starting place for generalizing the sine and cosine functions beyond the right triangle, but move quickly away from this representation.

To illustrate this inconsistent treatment, consider the initial definition of the sine and cosine functions. The presentation of these concepts may begin with the unit circle (as shown in Figure 2). The diagram treats angles as arising by rotating a radial line of length 1, emanating from the center of a circle centered at (0,0) on a (*x*,*y*) coordinate plane. Angles are measured in degrees or radians, with 360 degrees or 2π radians corresponding to a full rotation. The rotation is either counter-clockwise (for positive angles) or clockwise (for negative angles) from the starting position (where the radial line points horizontally from the origin to the right and corresponds to the line from (0,0) to (1,0)). The cosine of the angle is defined as the *x*-coordinate of the endpoint of the line, and the sine as corresponding to the *y*-coordinate. From this clearly visuospatial starting place, it is possible to shift almost entirely into algebra. For example, a fundamental trigonometric identity can be derived by applying the formula for the distance between two points (here 0,0 and the *x*,*y* coordinates of the endpoint of the radial line) and noting that, by the definition of the unit circle, this distance (the radius of the circle) is equal to 1. Applying the Pythagorian theorem, $x^2 + y^2 = 1$. Replacing x and y with $\cos\theta$ and $\sin\theta$ respectively, we obtain the identity $\cos^2\theta + \sin^2\theta = 1$. Furthermore, all of the other trigonometric functions can be defined algebraically (for example, $\tan\theta = \sin\theta/\cos\theta$; $\sec\theta = 1/\cos\theta$). Additional key identities, such as $1 + \tan^2\theta = \sec^2\theta$ may then be derived purely by algebraic manipulation or simply stated as rules to be learned, and the wave graphs of such functions (that is, graphs of the form $y = \cos\theta$ or $y = \sec\theta$) may then be introduced, with little or no engagement with the underlying construct that gives rise to the values of these functions. Thereby some of the features of these graphs (e.g. the fact that $\cos(0) = 1$ and $\sin(0) = 0$) are stripped of their relation to the underlying meaning of the functions and become arbitrary facts to be memorized by the student. Our experience with teachers in four pre-calculus classrooms at two high schools suggests considerable variability among teachers in their tendency to engage students with this or other visuospatial representations. While in some classrooms, whole 2-hour time blocks may be dedicated to teaching the unit circle, in others there is a tendency to rely on the teaching of rules or procedures that, if mastered, would allow students to calculate answers to particular problems without any comprehension of the expressions whatsoever.

What, indeed, are the best practices here? One can find passionately written articles such as the one by Shear (1985), a dedicated teacher of trigonometry, decrying the algebraic approach. Shear quotes Poincaré (1968) and Einstein (1979) for proclaiming their understanding of mathematical relationships in visuospatial terms, lays out a visuospatial framework in which the identities mentioned above are explicitly made visually apparent (or so he claims), and then asserts that use of his framework may allow students to gain "some sense of that elegance of thought and expression which, in the words of Poincaré, 'all true mathematicians know'". Unfortunately, Shear presents no evidence other than his own intuition and personal observations of his own students, and many textbook writers and classroom teachers continue to rely very heavily on the algebraic approach, especially when producing derivations. It is true that there is a textbook by Gelfand and Saul (1999) that develops a visuospatially grounded approach, but it is only one among many. It is also true that all teachers and all textbooks make use of some visuospatial examples (often, based on wave functions), but these uses vary widely in their extent and stated purpose and are not based on a scientific foundation.

Our effort to find scientific evidence has unearthed rather little beyond the above. We found one study based on a master's thesis (Kendal & Stacey, 1996) comparing two alternative visuospatial approaches to teaching right-triangle trigonometry problems of the kind covered in 9^{th} or 10^{th} grade geometry, as well as some more recent studies by Ninness and colleagues (Ninness et al., 2006; 2009) studying methods for teaching formula-to-graph relationships involving trigonometric functions and the ability to distinguish different types of relationships between formulas. These studies point to the difficulties many students have mastering the relationships between trigonometric functions and their graphs, and show that training using interactive graphing software can be effective, but the small sample size and lack of comparison conditions makes it difficult to draw inferences about the relative advantage of the approach. We also found a recent study by Moore (2013) applying the conceptual approach favored by Thompson et al. (1994) to helping students construct a cogent representation of the meanings of angular and radian measure in terms of arc length relative to the radius of a circle. Again only a small sample was used and there was no comparison of different approaches, but the approach is consistent with our perspective and informs our own studies.

PRELIMINARY INVESTIGATIONS

In accordance with the ideas that the use of a conceptual model within which symbolic expressions have meaning can contribute to success in mathematics and that reliance on such skills and models can be enhanced, we have undertaken a series of preliminary studies using undergraduate students at Stanford University. Our focus has been on a specific topic nearly all of these students were exposed to in high school, namely identities among trigonometric expressions involving the sine and cosine functions. We focused on a subset of such identities, involving the sine and cosine functions with positive or negative arguments between 10 and 80 degrees and subject to offsets of [–180, –90, 0, +90 or +180] degrees. While Stanford is a highly selective institution, not all Stanford undergraduates are especially strong in mathematics, allowing us to sample a range of prior experience and ability levels with respect to knowledge of these relationships. We asked three questions in our studies: (a) To what extent do students rely on visuospatial representations, as compared with rules or mnemonics, and among visuospatial representations, which do they use in solving the identity problems we used? (b) To what extent is the use of a particular representation associated with success in solving the class of problems we posed to our students? (c) For selected representations, how much can a student's performance be improved by a brief lesson based on a particular representation?

An example of the type of problem we used is shown in Figure 3. These problems all involved a probe expression, created by choosing either the sine or cosine function, followed by an expression in parentheses consisting of a signed two-digit base value and an offset (participants were told these values corresponded to degrees). There were 20 probe types, defined by all combinations of the function (sin or cos), the sign of the base value (positive or negative), and the value of the offset {–180, –90, 0, 90, 180}. Specific problems within each type further varied in the magnitude of the base value (equal to 20 in the example shown in the figure; always a multiple of 10 in the range [10, 80]), and order (base value first or offset first; when the first element was positive, its sign was omitted, as in this example). The four choice alternatives were always sin(θ), –sin (θ), cos(θ) and –cos(θ), where θ stands for the base value. One and only one alternative is correct for each problem. The participants were instructed to respond ‘quickly but still accurately’ and were asked not to use paper and pen/pencil or a calculator and not to refer to any outside sources.

How might problems of the kind described be solved? One approach would be to rely on rules and formulas. Indeed a very small number of rules, together with a few very general rules of algebra, are sufficient to solve all of these problems. For example the rule cos(θ+180) = –cos(θ) yields the correct answer in this case. A set of rules sufficient to answer all of the problems is given in Figure 4 (where **func** appears in this figure, it can be replaced with **sin** or **cos**; **opp** refers to **sin** if **func** refers to **cos** and *vice versa*).

In line with our hypotheses, we considered three visuospatial representations: (1) The unit circle; (2) the wave representation of the sine and cosine functions; and (3) the right triangle. These representations are all shown in Figure 4. As previously discussed, the unit circle treats angles as arising by rotating a radial line emanating from the center of a circle of radius 1 centered at (0,0) on a (x,y) coordinate system. Compound angles can be viewed as a sequence of rotations (e.g., 20 + 180 corresponds to rotating counterclockwise by 20 degrees and then continuing for another 180 degrees). The cosine of the overall angle can be visualized as a line segment extending either to the right (positive) or the left (negative) of the origin of the coordinate system. The sine can be visualized as a line segment extending either upward (positive) or downward (negative) from the origin. With this representation, the problems can be solved by comparing the directions and lengths of the line segments corresponding to the value of the probe and each of the choice alternatives. As shown on the unit circle diagram in Figure 4, for cos(–θ), the line segment corresponding to its value is the same as the line segment corresponding to the value of cos(θ), indicating that cos(–θ) = cos(θ).

In the wave representation, sine and cosine are represented as waves, each with a period of 360 degrees, with sin (lighter curve in Figure 4) shifted 90 degrees to the right relative to cos. The arguments of these functions in this representation specify positions along the *x* axis and compound arguments can be viewed as a sequence of translations from the 0 position; the value of the function can then be read off by examining the height of the wave at the appropriate point on the *x* axis (again, it can be seen that cos(–θ) = cos(θ)).

We also considered a third visuospatial representation, the right triangle. In such a triangle, the two acute angles must equal 90 degrees, the cosine of an angle is defined as the length of the adjacent side over the length of the hypotenuse, and the sine of an angle is defined as the length of the opposite side over the hypotenuse. Although this representation is not useful for all of our problems, it is useful for the cos(90–θ) or sin(90–θ) problems. For example, in the right triangle shown in Figure 4, the side adjacent to the angle θ is the side opposite the angle (90–θ) so that cos(θ) and sin(90–θ) both refer to the ratio of the length of the same leg of the right triangle (the horizontal leg) to the hypotenuse; in other words, sin(90–θ) = cos(θ). In addition, we explored the possibility that participants might rely on mnemonics, such as SOHCAHTOA ("sine equals opposite over hypotenuse, cosine equals adjacent over hypotenuse, tangent equals opposite over adjacent") and All Students Take Calculus. These mnemonics combine rules and visual representations as illustrated for All Students Take Calculus in the lower right of Figure 4.

Preliminary Study: Observing Use and Success of the Unit Circle

In our preliminary study, students encountered two examples of each of the 20 problem types in each of two blocks of trials. At the end of the experiment, we asked participants to rate how frequently they used each of the five possible representations on a scale from 1-5 (1=Never, 5=Always; a sixth 'other representation' category was included but rarely used). Ratings were given first for block 1 and then for block 2 of the experiment. We then asked them to report the number and recency of courses in which they had used trigonometry. One group of participants (n = 12) had a brief break between blocks, while the remaining two groups had a brief lesson, either involving rules (n = 12) or waves (n = 13). Only ratings of block 1 were used in analyses.

Overall performance varied widely, with the median score falling at 66.3% correct and several participants performing at near-chance levels (all participants appeared to be trying to respond correctly and nearly all did well on the very easy problems of the form $\cos(\theta+0)$ and $\sin(\theta+0)$). Order of arguments in parentheses (base or offset first) did not affect responding.

Neither the rule lesson nor the wave lesson led to greater improvement in the second block compared to the no-lesson control condition.

The unit circle representation was generally favored by our participants. Overall, participants reported using the unit circle more frequently (Mean 3.5) than any other representation (largest other Mean, 2.4). Furthermore, we found a positive correlation between extent of reported unit circle use and performance score (proportion of problems correct) that remained a significant independent predictor of performance after taking into account both number of courses and recency.

Exploratory analysis revealed a striking pattern on one subset of problems (see Figure 5), namely problems involving the sine or cosine of a negative argument and an offset of 0, i.e. $\sin(-\theta+0)$ and $\cos(-\theta+0)$. Most participants answered $\sin(-\theta+0)$ problems correctly (choosing $-\sin(\theta)$), regardless of use of the unit circle. For $\cos(-\theta+0)$, however, the correct answer is not $-\cos(\theta)$ but simply $\cos(\theta)$ (As noted above, Figure 4 shows this on the unit circle). In this case, participants who reported 'Always' using the unit circle ($n = 15$) answered this class of problems correctly 83% of the time, whereas other participants ($n = 22$) chose the correct answer only 36% of the time. The predominant error for these participants was $-\cos(\theta)$, a response these participants made more frequently than the correct response.

These results provide promising evidence for a positive role of visualizing the unit circle in solving trigonometric identity problems of the kinds that we used, but are limited in several ways. The ratings were global and retrospective, and the presence of a lesson may have altered the representations participants used or their recollections of these representations (though this should, if anything, have reduced reports of unit circle use relative to rules or waves). More importantly, the groups were too small to meaningfully assess the effect of the lessons, and the lessons did not include a circle lesson, preventing a causal inference about its usefulness.

Study 2: Comparing a Unit Circle Lesson to a Rules Lesson and Baseline Knowledge

Two further studies addressed these limitations. The first (Study 2.1) followed up on the relationship between unit circle use and performance in a larger group of participants (N = 50) without any lesson. The experiment followed the same protocol as in the no-lesson condition of the preliminary study, avoiding influence of a lesson on retrospective ratings (we did not solicit ratings during performance of either of the two main blocks of trials to avoid suggesting strategies to participants). To supplement the retrospective ratings, an additional block of 20 problems was added. After selecting a response for each of these problems, the participants rated their use of each representation on that problem on a three point scale (1 = not at all, 2 = a little, 3 = a lot). Participants then answered further questions about the extent of their prior exposure to and use of each of the five representations, and finally participated in a short videotaped session in which the experimenter interviewed each participant on their strategy in solving one problem of each of the three types illustrated in Figure 4. In the second further study

(2.2), participants also completed the same protocol, with the modification that they received either a lesson on the unit circle (*n* = 35) or in the use of rules and formulas (*n* = 35) between the first and second block. Lessons covered the inputs (signed argument plus offset) and outputs (resulting values) of the sine and cosine functions, and specifically described how either the unit circle or a simple rule could be used to correctly solve problems of the forms cos(θ+180), sin(θ+180), cos(–θ+0), sin–θ+0), sin(θ+90), cos(θ+90).

Study 2.1 replicated and extended all of the findings from the preliminary study. Average retrospective ratings of representations used in both the first and second block once again showed greater rated use of the unit circle compared to any of the other representations, and once again circle use rating strongly covaried with overall performance, even after controlling not only for number of courses and recency, but also for prior unit circle exposure and prior unit circle use. That is, controlling for prior unit circle exposure and use, higher reported use in our experiment predicted higher overall performance.

In a now-planned comparison, we replicated the pattern of findings on the relation between average retrospective rated circle use and performance on cos(–θ+0) during blocks 1 and 2. The pattern was largely the same as before, except that among those who said they always used the unit circle in their retrospective ratings, there were fewer correct responses than in the preliminary study, and the predominant error was –cos(θ), suggesting that some participants who stated that they relied on the unit circle either lacked sufficient mastery or imagery of the unit circle construct, or perhaps did not use the unit circle on these specific problems. Our problem-specific ratings allowed us to follow up on this pattern. Here we found that those who reported using the unit circle 'a lot' on problems of this type averaged 75% correct and did not tend to make the $-\cos(\theta)$ error on such problems during blocks 1 and 2.

In summary, Study 2.1 supports the conclusion that unit circle use is generally associated with more accurate performance, but requires more time than some other strategies. Some who say they always use the unit circle in retrospective ratings do less well than others, but both response times and problem-specific circle ratings suggest that most of these participants do not rely on the unit circle for $\cos(-\theta+0)$ problems. Instead, we hypothesize that these and many other participants may rely on a faulty rule or heuristic corresponding to 'pulling out the minus sign' from both the sin and cos functions. This error reflects a lack of engagement with the underlying meaning of these functions. It takes less time than visualizing the functions on the circle, and gives the correct answer for $\sin(-\theta+0)$, but fails for $\cos(-\theta+0)$. A small number of relatively expert users may know and understand correct rules that allow them to by-pass constructing a visuospatial representation at least for certain problems. Others may attempt to construct a visuospatial representation but fail to do so with sufficient reliability.

The results of our follow-up training Study 2.2 provide further support for the value of the unit circle. The unit circle lesson led to more improvement from block 1 to block 2 than the rule lesson. Since the participants in both Study 2.1 and 2.2 were drawn from the same pool and since their experiences were identical through the end of the second block except for the presence of a lesson, we used the participants in Study 2.1 as a no-lesson control, comparing the improvement from block 1 to block 2 shown by these participants to the improvement exhibited by each of the two lesson groups from Study 2.2. We also compared problem types directly taught in the lessons with problems that were not directly taught, hereafter called transfer problems. (The trivial cos(θ+0) and sin(θ+0) problems were excluded from this analysis). This analysis revealed that both lessons helped participants with problems that were directly taught, but participants who received the unit circle lesson benefitted on transfer problems (16% gain) compared to controls (6% gain), while participants who received the rule lesson did not show a benefit on transfer problems (3% gain) compared to controls.

A further analysis considered whether the improvement in performance shown by participants who received the circle or rule lesson was moderated by their pre-test performance. We examined the overall improvement (combined over taught and transfer items) by those whose pretest performance was 'near chance' (within the 95% confidence interval of 25% correct on non-trivial problems), vs. those whose pre-test performance was above the near chance level but below a 95% correct ceiling level. Among participants receiving the unit circle lesson, those in the above chance group benefited more overall than those in the near chance group; for those in the rule lesson condition, both groups showed a similar, small overall benefit compared to corresponding controls. (Also, after controlling for pre-test performance, there was no significant sex difference in gains after either lesson.) Caution is required in interpreting this pattern since it is based on an unplanned analysis. That said, the pattern strongly resonates with a large body of other work in which improvement based on experience depends on the participant having a foundation (perhaps implicit) on which the experience can build, spanning fields as diverse as development of an intuitive understanding of balance (Siegler, 1976) and the spatial memory formation in rodents (Tse et al., 2007, 2011), as well as our own modeling work addressing these phenomena (McClelland, 1995; McClelland, 2013).

In summary, we found that participants exhibit a generalizable performance gain after a brief lesson based on the unit circle, relative to those receiving no lesson or those receiving a lesson based on rules. Importantly, we also found that the ability to benefit from the lesson may depend on the participants' being in a state of readiness to learn (Siegler, 1976) or what Vygotsky (1978) called the 'zone of proximal development' (Chaiklin, 2003). In the context of the grounded conceptual structures framework described in the introduction, one can potentially understand what readiness means in terms of the availability of sufficient degree of prior establishment of the relevant structures and their constituents, a point to which we will return. Given this issue and other limitations of these studies, it will be important to examine these issues further.

CHALLENGES IN LEARNING THE UNIT CIRCLE

The research we have begun to undertake resonates with calls over many years to enhance the reliance on use of spatial representations that instantiate the meanings of mathematical expressions in visualizable form rather than rote memorization of rules (Polya, 1945; Shear, 1985, Thompson et al., 1994; Wertheimer, 1959). While all trigonometry teachers and texts connect with visuospatial representations to some degree, they do so to strikingly varying extents, and in different, apparently *ad hoc* ways. By conducting studies aimed to provide evidence about which types of representations are most useful for solving problems of different types by students of varying ability levels, our work may pave the way to establishing the evidence needed to begin to construct evidence-based teaching programs.

Our principal goal is to further assess, and potentially establish a broader basis of support for, the proposition that grounding pre-calculus trigonometry instruction on a meaningful visuospatial representation can contribute substantially to student learning of this material. This entails further assessing the relative merits of teaching students to rely on such a representation relative to a rule-based lesson. In order to do this, we intend to correct two limitations of Study 2.2: (1) the fact that the study was undertaken with a group of students at an elite university, almost all of whom reported having prior experience or exposure to the relevant ideas and concepts and (2) the relatively small size of the effect of the lessons, especially among students who performed relatively poorly in the pre-test. In an informal replication of the unit circle lesson condition from Study 2.2, administered to 17 pre-calculus students at Palo Alto High school, all but one of these students fell into the 'near chance' range of our previous study, even though they were just coming to the end of an eight-week instructional block that covered the relevant material. Like our Stanford students, these students systematically erred on the $\cos(-\theta+0)$ problem, with 70% incorrectly choosing $-\cos(\theta)$. Furthermore, the gain in performance from pre-test to post-test was even more modest (6%) than it was for the near-chance pretest subgroup of the Stanford group. It should be noted that there are several faster-paced 'lanes' at Palo Alto High School, and those who participated largely came from the middle or bottom thirds of their classroom cohorts; thus, their performance relative to Stanford students should not necessarily be surprising. Nevertheless, these results underscore the need to find ways of improving mastery of pre-calculus concepts for middle-of-the-road students, to enhance their ability to move into STEM careers. These results also point to the challenge we face in developing a lesson that can address the needs of these students.

As we work toward the eventual goal of addressing this challenge, our work will use community college and high-school students who have not already taken mathematics courses beyond Geometry and Algebra 2. Work of other researchers suggests that college students encountering pre-calculus relevant mathematical content for the first time experience similar difficulties as high-school students (Moore & Carlson, 2012; see also Carlson, Oehrtman, & Engelke, 2010), and ready access to a large population of such students makes this an attractive next step for our efforts.

Unit circle instruction for students without prior pre-calculus trigonometry

The lesson used in the studies discussed above was designed with Stanford undergraduates in mind, and in order to be more effective for students with less prior experience, we hypothesize that it must be adapted in several ways. In keeping with the idea that the emergence of a coherent, grounded conceptual structure is a gradual process, the new versions of our lessons will be longer, allowing for repetition and practice using the material covered in each lesson. We will incorporate active visuospatial responding with the mouse or trackpad (constructing angles and projections in the unit circle lesson) or manipulating symbolic expressions at the keyboard (in the rule-based lesson); this is likely to increase the mental engagement during learning (compared with the requirement to simply read the material as in the existing lesson) and therefore enhance learning (Ninness et al. 2009; Schmidt & Bjork, 1992).

Rather than needing more repetition or enhancement, perhaps some students do not have the appropriate prerequisite knowledge or experience in order to be ready to learn the unit circle. The unit circle provides a conceptual structure that integrates two component structures (the *x,y* plane and the circle) into one, and those components that are assumed to be familiar already may not be readily accessible to a student. In order to use the unit circle, one must first comprehend an angle and be able to mark the endpoint of a radius representing one side of that angle on a circle. Then, one must be able to add or subtract positive and negative angles, or mentally visualize a compound angle as the appropriate sequence of rotations. Then, to relate the resulting endpoint of a radial line to its sine or cosine, a student must project the endpoint onto the appropriate axis of the coordinate plane to find a number between –1 and 1. These skills should have been acquired with relevant experience in previous mathematical settings, but may not be fully mastered by all students. Students must have a high fluency with these steps as they are embedded in the context of problems that are presented as symbolic expressions such that any reliance on visuospatial representation would require participants to form and manipulate the visuospatial representation in their minds. We also note that this sequence of procedures may recruit general visuospatial abilities that may be malleable (Uttal et al., 2013) like those used in mentally rotating block shapes (Shepard & Metzler, 1971; Vandenberg & Kuse, 1978) or in finding simple shapes or parts in more complex figures (Ekstrom, French, Harman, & Derman, 1976).

We presented a new and considerably extended set of materials intended to address the points above to a small pilot group of eleventh graders near the beginning of their exposure to pre-calculus trigonometry, and found that, although some students had initial misconceptions, they progressed easily through the lesson sessions focused on knowledge prerequisites for the unit circle: the number line, the coordinate plane, and angles rotations of a radial line around the circle. They showed (and self-reported) greater difficulty in fully gaining an understanding of how the sine and cosine functions linked the position of the endpoint of a radial line corresponding to an angle to the endpoints' *x* and *y* coordinates, and they encountered additional difficulty when required to compare values of expressions involving the sine and cosine functions when these were presented symbolically without the availability of an external unit circle. These difficulties suggest that our lesson may need to be adapted to allow more practice internalizing a representation of the meaning of the sine and cosine functions and also allowing the transition from reliance on an external representation of the unit circle to an internalized representation of it to occur more gradually. We also suspect the materials may need to be adapted to address counterproductive epistemic beliefs and mindsets some students may bring to trigonometry learning. We consider these issues in turn in the next two sections.

Internalizing the Unit Circle

To facilitate the development of a sufficiently internalized and integrated understanding of the unit circle, we will adapt the *Externalization then Internalization* approach of Glenberg et al. (2004). In our unit circle lesson, students will initially be taught their way around the basic conceptual structure of the unit circle by constructing angles initially under carefully guided instruction and with external supports that will gradually be removed. For example, students will be introduced to positive and negative angles, and given opportunities to construct such angles by rotating lines around a unit circle; they will initially have critical landmark angles labeled, but these labels will then be gradually removed. Students will then be introduced to compound angles, and will be given opportunities to construct compound angles by successive rotations corresponding to each of the compound angle's parts. They will be given the opportunity to construct equivalent resulting angles by carrying out the corresponding rotations in different orders, and observing the end result.

Once the concepts of sin and cos are introduced, students will at first be asked to position a radial line to correspond to a specific example angle and then to construct the projection of the endpoint of the radial line onto the appropriate axis to represent the angle's sine or cosine value. We will gradually eliminate reliance on the externalization, replacing it with the instruction to carry out the corresponding operations mentally, since this instruction was found necessary to produce robust comprehension gains in Glenberg et al. (2004). Rather than eliminating external support all at once, we plan to do so in stages. For example, one intermediate step might be to ask students to construct an approximate representation of the sine or cosine of a particular angle (as points or segments on the x-y coordinate plane) without externally positioning the radial line corresponding to the angle. After repeated practice along this progression, students will be better prepared to transition to comparison of the values of two trigonometric expressions. Here again, we will first allow them to explicitly construct the value of each expression externally, one on each of a pair of side-by-side copies of the unit circle, before gradually removing these external aids and asking the students to perform approximations of these comparisons completely 'in their heads'. One intriguing possibility that we are exploring is that the ability to make these comparisons mentally may rely on internalization of schematic representations that do not represent specific angles exactly but do represent relationships between (for example) the projection onto the x axis of the endpoint of a radial line corresponding to positive and negative rotations of the same amount. One may not be able to visualize a particular given angle correctly, but one may be able to 'see' that the endpoints of both the positive and negative rotations of the same amount project to the same point on the x axis through the unit circle.

The role of epistemic belief in acquiring an integrated conceptual representation

We are also exploring the roles of what others have called students' epistemic beliefs and mindsets in enabling their ability to acquire the unit circle framework as an integrated conceptual structure. If students believe the goal of our lesson is to gain an understanding of trigonometry, what do they believe is the nature of such an understanding? The unit circle provides a conceptual framework that supports an integrated problem solving procedure that can be applied across a range of problem types, whereas rules are narrower, with each one applying only to a limited set of problem types. If students think that understanding mathematics is a matter of learning a set of unrelated rules, they may struggle when confronted with new types of problems not covered by existing rules (like the transfer problems in our task). In a study on learning statistics, students who believed mathematics consisted of a set of simple facts tended to show worse comprehension than those who saw mathematics as an integrated system of relationships (Schommer, Crouse, & Rhodes, 1992). These are examples of what the authors called epistemic beliefs.

Other beliefs they call epistemic include beliefs about one's innate ability to learn, the quickness of learning, and the source and nature of knowledge (Schommer, 1990). Such beliefs may be malleable: students who were encouraged to believe that intelligence is a malleable quality (displaying a "growth mindset") showed greater motivation as well as stronger grades in their mathematics classes (Blackwell, Trzesniewski and Dweck, 2007). Students with a growth mindset also more strongly endorsed the idea that effort is necessary to learn, which may play a role in expectations about the quickness of learning and thus the ability of students to be patient and take the necessary time to learn, rather than rushing through materials for the sake of getting through them. Because these factors seem likely to us to have important influences on a student's ability to master the unit circle as an integrated conceptual structure underlying trigonometry, the next version of our materials will explicitly encourage viewing trigonometry as in integrated system of knowledge that requires time and engagement to learn. The materials will be introduced as an integrated framework as the students start into the program of lessons, and each block of the lesson will described as playing a specific role within an integrative approach to understanding the meaning of trigonometric expressions within the unit circle framework.

CONCLUSIONS AND FUTURE DIRECTIONS

In this chapter, we have argued that a particular visuospatial model called the unit circle acts as a grounded conceptual structure in trigonometry. In preliminary studies, we found that students who reported using the unit circle performed better at solving trigonometry identities than those who did not report visualizing this representation. We also found that a brief lesson in the use of the unit circle produced post-lesson benefits relative to no lesson or a rule-based lesson. These benefits were seen in participants who exhibited some partial understanding of the relevant concepts in a pretest, whereas high school students who lacked facility with components of the unit circle had more difficulty learning in a unit circle lesson. The difficulty students without sufficient prior knowledge experienced illustrates the challenges we face in helping students learn this conceptual structure.

We have described some of the steps we are taking to address the challenges we have identified. First, we hope to strengthen student's command of the components that are integrated in the unit circle. Second, we hope to foster their gradual internalization, so that when students confront expressions in trigonometry, they can visualize the component elements and their relationship to each other mentally, rather than relying on an external visuospatial representation. In addition, we plan to measure learners' initial epistemic beliefs and mindsets and to provide explicit encouragement of beliefs that will, we hope, encourage the learner to understand that the goal is to build an integrated conceptual structure and that to do so may require patience and practice. Finally, our future work will also explore how individual differences in visuospatial or other aspects of cognitive abilities affect students' acquisition and use of the unit circle representation. As educators and policymakers examine the structure and content of the trigonometry curriculum, we hope our work will eventually lead to the development of effective practices for allowing learners to acquire an integrated conceptual framework for understanding trigonometry.

References

Bartlett, F. C. (1932). *Remembering.* Cambridge: Cambridge University Press.

Barwise, J., & Etchemendy, J. (1996). Visual information and valid reasoning. In G. Allwein & J. Barwise (Eds.), *Logical reasoning with diagrams* (pp. 3-26). New York: Oxford University Press.

Blackwell, L. S., Trzesniewski, K. H., & Dweck, C. S. (2007). Implicit theories of intelligence predict achievement across an adolescent transition: A longitudinal study and an intervention. *Child Development*, *78*(1), 246-263.

Bransford, J. D., Barclay, J. R., & Franks, J. J. (1972). Sentence memory: A constructive versus interpretive approach. *Cognitive Psychology*, *3*(2), 193-209.

Bransford, J. D. & Johnson, M. K. (1972). Contextual prerequisites for understanding: Some investigations of comprehension and recall. *Journal of Verbal Learning and Verbal Behavior*, *11*(6), 717-726.

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment (PCA) instrument: A tool for assessing reasoning patterns, understandings, and knowledge of precalculus level students. *Cognition and Instruction, 28*(2), 113-145.

Case, R. (1996). Introduction: Reconceptualizing the nature of children's conceptual structures and their development in middle childhood. *Monographs of the Society for Research in Child Development, 61*(1-2), 1-26.

Chaiklin, S. (2003). The Zone of Proximal Development in Vygotsky's analysis of learning and instruction. In Kozulin, A., Gindis, B., Ageyev, V. & Miller, S. (Eds.), *Vygotsky's educational theory and practice in cultural context* (pp. 39-64). Cambridge: Cambridge University.

Dantzig, T. (2007). *Number: The language of science* (Masterpiece Science ed.). New York: Penguin Group.

Dehaene, S. (1992). Varieties of numerical abilities. *Cognition*, 44, 1-42.

1-42.Einstein, A. (1979). *Ideas and opinions.* New York: Dell Publishing Co.

Ekstrom, R. B., French, J. W., Harman, H. H., & Derman, D. (1976). *Manual for Kit of Factor-Referenced Cognitive Tests.* Princeton, NJ: Educational Testing Service.

Foerster, P. A. (1990). *Trigonometry: Functions and applications* (3rd ed.). Reading, MA: Addison Wesley Publishing.

Gelfand, I. M. & Saul, M. (2001). *Trigonometry.* Boston, MA: Brinkhauser.

Glenberg, A. M., Gutierrez, T., Levin, J. R., Japuntich, S., & Kaschak, M. P. (2004). Activity and imagined activity can enhance young children's reading comprehension. *Journal of Educational Psychology, 96*(3), 424-436.

Goldstone, R. L. & Son, J. Y. (2005). The transfer of scientific principles using concrete and idealized simulations. *Journal of the Learning Sciences*, *14*(1), 69-110.

Harnad, S. (1990). The symbol grounding problem. *Physica D: Nonlinear Phenomena, 42*(1), 335-346.

Hornsby, E. J., Lial, M. L., & Rockswold, G. K. (2011). *A graphical approach to precalculus with limits: A unit circle approach* (5th ed.). Boston, MA: Addison-Wesley.

Johnson-Laird, P. N. (1983). *Mental models: Towards a cognitive science of language, inference, and consciousness*. Cambridge, MA: Harvard University Press.

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). The advantage of abstract examples in learning math. *Science, 320*(5875), 454-455.

Kell, H. J., Lubinski, D., Benbow, C. P., & Steiger, J. H. (2013). Creativity and technical innovation spatial ability's unique role. *Psychological Science*, *24*(9), 1831-1836

Kelly, D., Nord, C. W., Jenkins, F., Chan, J. Y., & Kastberg, D. (2013). Performance of US 15-Year-Old Students in Mathematics, Science, and Reading Literacy in an International Context. First Look at PISA 2012. NCES 2014-024. *National Center for Education Statistics*.

Kendal, M., & Stacey, K. (1996). Trigonometry: Comparing ratio and unit circle methods. In P. C. Clarkson (Ed.), *Technology in mathematics education* (Proceedings of the 19th annual conference of the Mathematics Education Research Group of Australasia). Melbourne: MERGA.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic books.

Lewis, A. B. (1989). Training students to represent arithmetic word problems. *Journal of Educational Psychology, 81*(4), 521-531.

Marcus, G. F. (2003). *The algebraic mind: Integrating connectionism and cognitive science*. Cambridge, MA: The MIT Press.

Mayer, R. E., & Hegarty, M. (1996). The process of understanding mathematical problems. In R. J. Sternberg & T. Ben-Zeev (Eds.), *The nature of mathematical thinking* (pp. 29-53). New York: Routledge.

McClelland, J. L. (1995). A connectionist approach to knowledge and development. In T. J. Simon & G. S. Halford (Eds.), *Developing cognitive competence: New approaches to process modeling.* pp. 157-204. Mahwah, NJ: LEA, 157-204.

McClelland, J. L. (2013). Incorporating rapid neocortical learning of new schema-consistent information into complementary learning systems theory. *Journal of Experimental Psychology: General, 142*(4), 1190-1210.

Moore, K. C. (2013). Making sense by measuring arcs: a teaching experiment in angle measure. *Educational Studies in Mathematics, 83*(2), 225-245.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior, 31*(1), 48–59.

Moss, J. & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*(2), 122-147.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. US Department of Education.

Newell, A. & Simon, H. A. (1961). Computer simulation of human thinking. *Science, 134*(3495), 2011-2017.

Ninness, C., Barnes-Holmes, D., Rumph, R., McCuller, G., Ford, A. M., Payne, R., …, & McGinty, J. (2006). Transformations of mathematical and stimulus functions. *Journal of Applied Behavior Analysis, 39*(3), 299-321.

Ninness, C., Dixon, M., Barnes-Holmes, D., Rehfeldt, R. A., Rumph, R., McCuller, G., …, & McGinty, J. (2009). Constructing and deriving reciprocal trigonometric relations: A functional analytic approach. *Journal of Applied Behavior Analysis, 42*(2), 191-208.

Organization for Economic Cooperation and Development (OECD). (2004). *Learning for Tomorrow's World - First Results from PISA 2003*. Paris: Author.

Poincaré, J. H. (1968). Mathematical Creation. In M. Kline (Ed.), *Mathematics in the Modern World: Readings from Scientific American*. San Francisco: W. H. Freeman & Co.

Polya, G. (1945). *How to solve it.* Princeton, NJ: Prinseton University Press.

Russell, B. (1903). *The principle of mathematics*. Cambridge: University Press.

Schmidt, R. A., & Bjork, R. A. (1992). New Conceptualizations of Practice: Common Principles in Three Paradigms Suggest New Concepts for Training. *Psychological Science, 3*(4), 207-217.

Schommer, M. (1990). Effects of beliefs about the nature of knowledge on comprehension. *Journal of Educational Psychology*, *82*(3), 498-504.

Schommer, M., Crouse, A., & Rhodes, N. (1992). Epistemological beliefs and mathematical text comprehension: Believing it is simple does not make it so. *Journal of Educational Psychology*, *84*(4), 435-443.

Shear, J. (1985). Visual thinking, algebraic thinking, and a full unit-circle diagram. *The Mathematics Teacher*, *78*(7), 518-522.

Shepard, R. N. (2008). The step to rationality: The efficacy of thought experiments in science, ethics, and free will. *Cognitive Science*, *32*(1), 3-35.

Shepard, R. N., & Chipman, S. (1970). Second-order isomorphism of internal representations: Shapes of states. *Cognitive psychology*, *1*(1), 1-17.

Shepard, R. N., & Metzler, J. (1971). Mental rotation of three dimensional objects. *Science,* *171*(3972), 701-703.

Siegler, R. S, (1976). Three aspects of cognitive development. *Cognitive Psychology*, *8*(4), 481-520.

Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. A. (1994). Calculational and conceptual orientations in teaching mathematics. In A. Coxford (Ed.), *1994 Yearbook of the NCTM* (pp. 79-92). Reston, VA: NCTM.

Tse, D., Langston, R. F., Kakeyama, M., Bethus, I., Spooner, P. A., Wood, E. R., ... & Morris, R. G. (2007). Schemas and memory consolidation. *Science, 316*(5821), 76-82.

Tse, D., Takeuchi, T., Kakeyama, M., Kajii, Y., Okuno, H., Tohyama, C., ... & Morris, R. G. (2011). Schema-dependent gene activation and memory encoding in neocortex. *Science, 333*(6044), 891-895.

Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., & Newcombe, N. S. (2013). The malleability of spatial skills: A meta-analysis of training studies. *Psychological Bulletin*, *139*(2), 352-402.

Vandenberg, S. G. & Kuse, A. R. (1978). Mental rotations, a group test of three-dimensional spatial visualization. *Perceptual and Motor Skills, 47*(2), 599-604.

Vygotsky, L. S. (1978). Interaction between learning and development (M. Lopez-Morillas, Trans.). In M. Cole, V. John-Steiner, S. Scribner, & E. Souberman (Eds.), *Mind in society: The development of higher psychological processes* (pp. 79–91). Cambridge, MA: Harvard University Press.

Wai, J., Lubinski, D., and Benbow, C. P. (2009). Spatial Ability for STEM Domains: Aligning Over 50 Years of Cumulative Psychological Knowledge Solidifies Its Importance. *Journal of Educational Psychology, 101*(4), 817-835.

Wertheimer, M. (1959). *Productive thinking* (Enlarged Ed.). New York: Harper & Brothers.

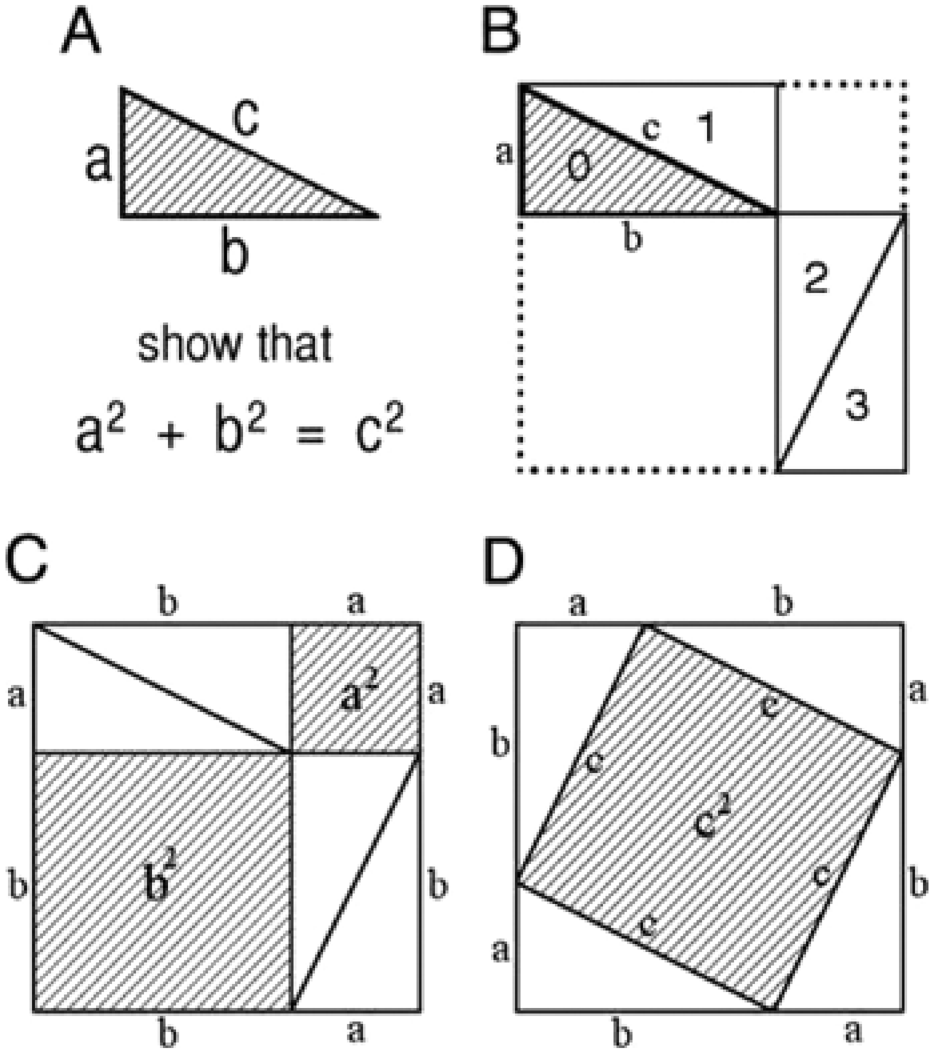


Figure 1. A proof of the Pythagorean theorem. Figure 1 from "The Step to Rationality: The Efficacy of Thought Experiments in Science, Ethics, and Free Will" by R. N. Shepard (2008), *Cognitive Science,* 32 (1), p. 6. Copyright 2008 Cognitive Science Society, Inc. Reprinted with permission pending.

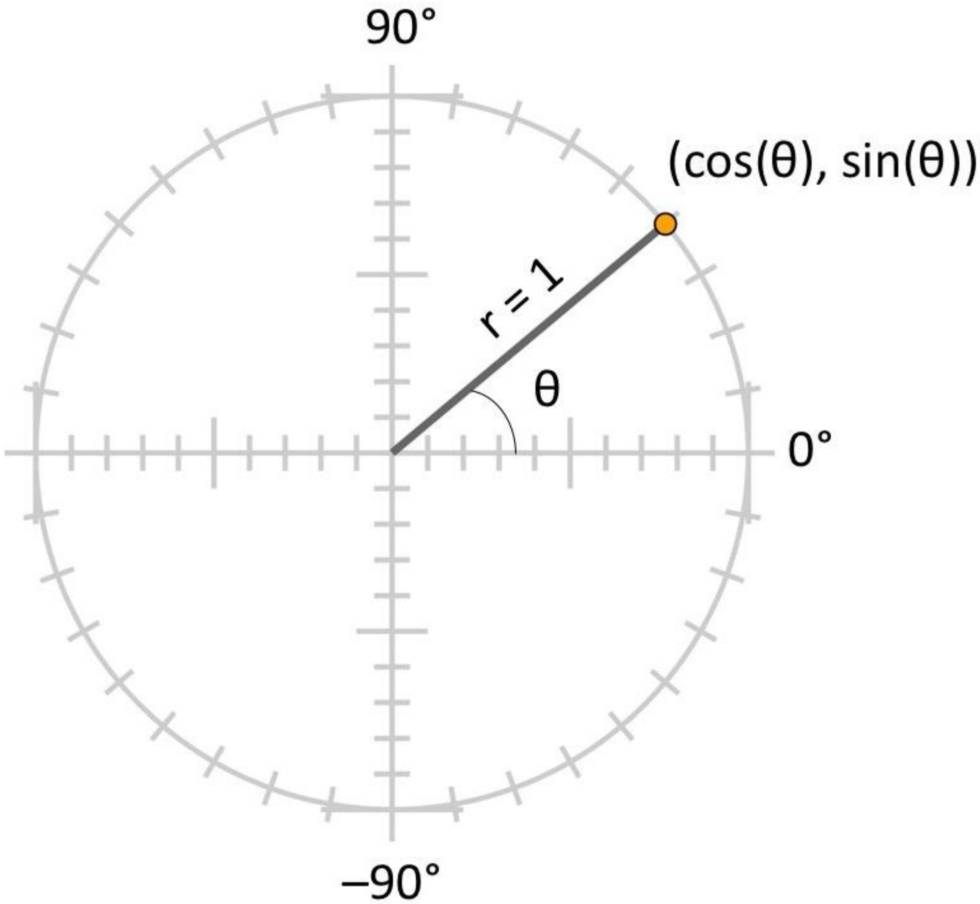


Figure 2. The unit circle.

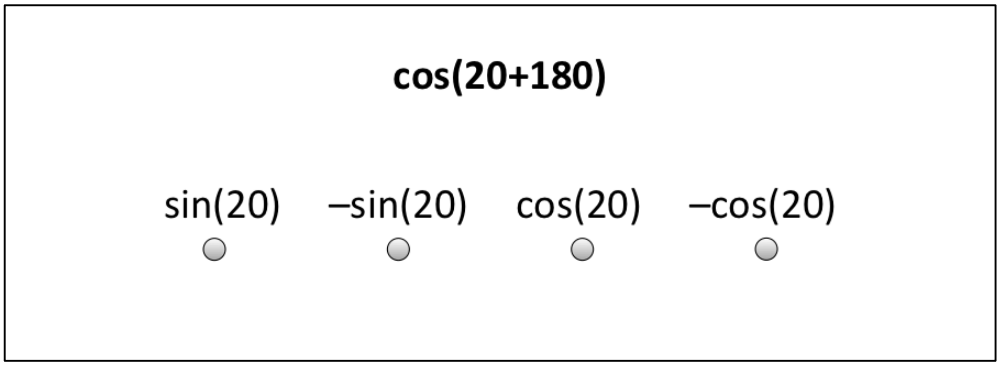
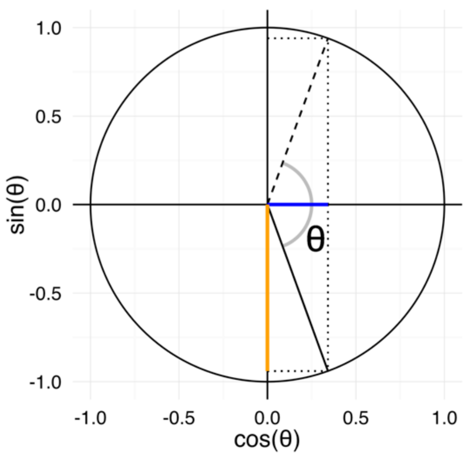


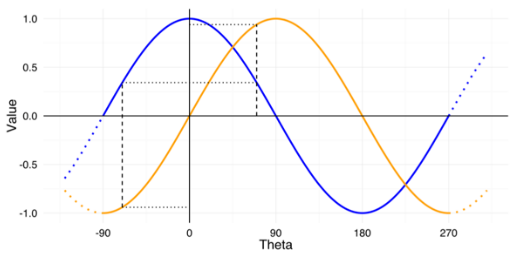
Figure 3. An example problem. Participants saw a probe expression at the top of an on-screen display and were instructed to choose the equivalent expression from the alternatives below it.

Spatial Representations

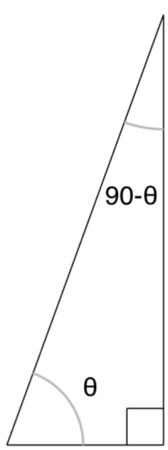
Unit Circle



Waves



Triangle



Symbolic Rules

Trigonometric Identities

sin(−θ) = −sin(θ)
cos(−θ) = cos(θ)
func(θ±180) = −func(θ)
func(90−θ) = opp(θ)

Principles of Algebra

x ± 0 = x
−(−x) = x
−x−y = −(x+y)
x+y = y+x

A Mnemonic

Symbolic Rule Component

Mnemonic	Positive functions	In quadrant:
All	**A**ll	I
Students	**S**ine	II
Take	**T**angent	III
Calculus	**C**osine	IV

Visuospatial Component

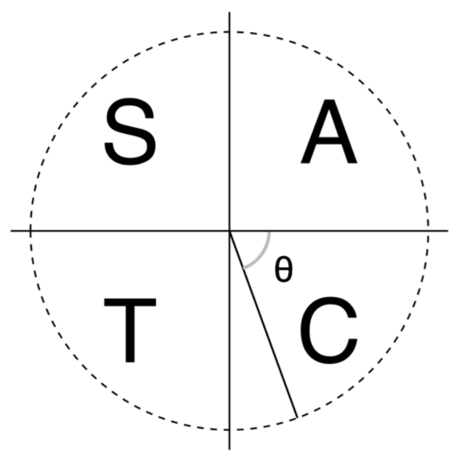


Figure 4. Some representations used in the domain of trigonometry. In the rules shown, ***func*** could be **sin** or **cos**; when ***func*** refers to one of these, ***opp*** refers to the other.

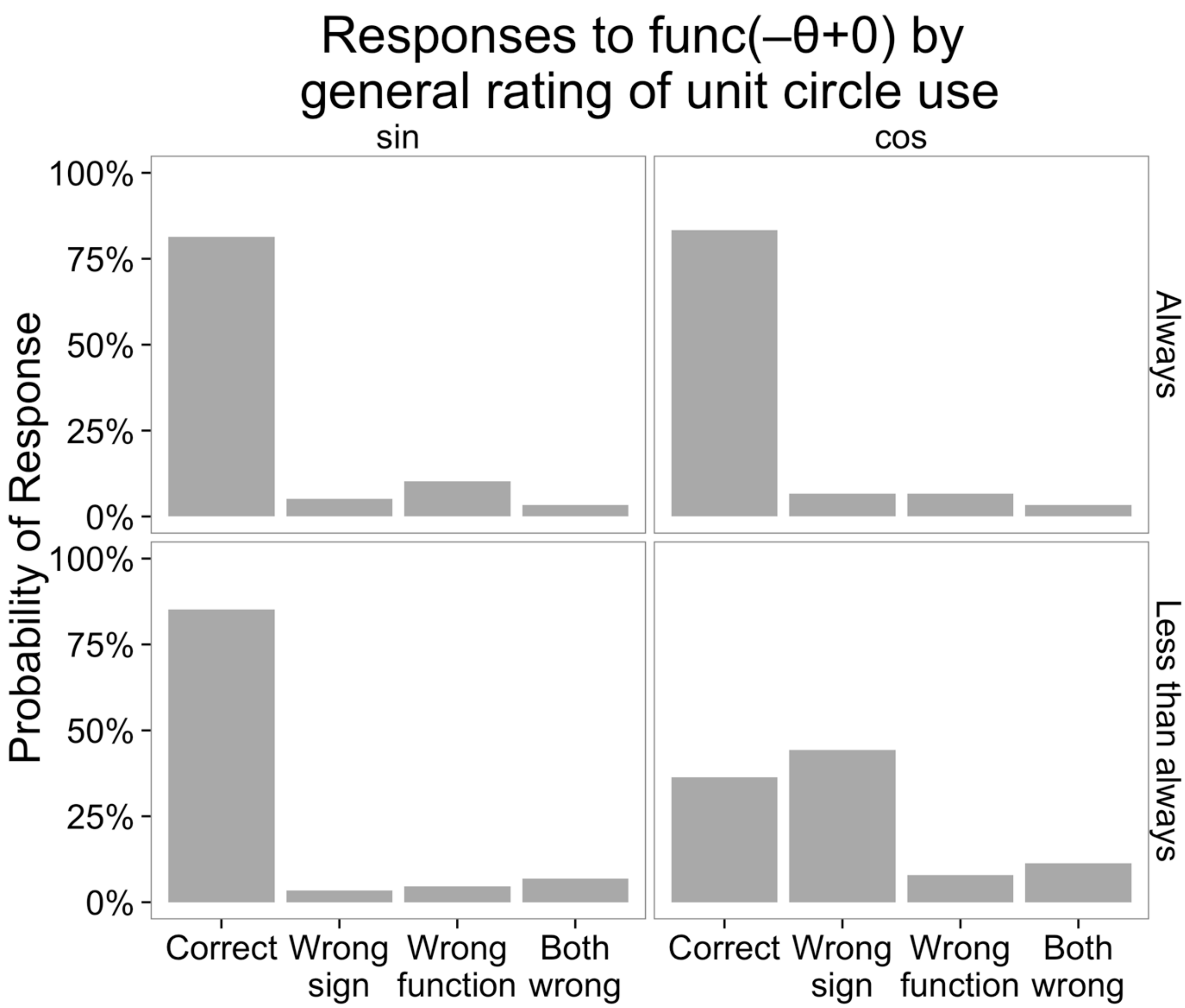


Figure 5. The distribution of responses to func(–θ+0), split by function and by general rating of unit circle use.